EXERCISES

Question NO : 1

Consider the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$IC : u(x, 0) = sin(\pi x), 0 < x < l,$

$BC's : u(0, t) = 0, u(l, t) = 0, t > 0$

1. Find explicit solution by method of separation of variables.
2. Apply Crank- Nicholson method and check stability and consistency.
3. Compute numerical solution for different values of h and k.

ANSWER :

(i)

GIVEN:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{0.0.1}$$

Assume that $u(x,t) = X(x)T(t)$.

$\Rightarrow u_x = X'T, u_{xx} = X''T, u_t = X\dot{T}$ where $X' = \frac{dX}{dx}, \dot{T} = \frac{dT}{dt}$

Then () becomes

$$X\dot{T} = X''T$$

$$(ie) \quad \frac{\dot{T}}{T} = \frac{X''}{X} = k \; (independent\ of\ t\ and\ x)$$

Suppose $k > 0$.

Then the solution of the equations

$X'' = kX$ is given by

$X(x) = c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x}$

Since $u(0,t) = u(l,t) = 0$. we get

$x(0) = c_1 + c_2 = 0$ and $X(l) = c_1 e^{\sqrt{k}l} + c_2 e^{-\sqrt{k}l} = 0$

$\Rightarrow c_1 = c_2 = 0$.

$\therefore X(x) = 0$.

$\Rightarrow u(x,t)=0.$

$\therefore$ The given problem has no solution if $f\neq 0$.

Take $k=-\lambda^2$,

Then the solution of the equation

$X''=-\lambda^2 X$ is given by

$X(x)=c_1\cos(\lambda x)+c_2\sin(\lambda x)$.

$$\begin{aligned}X(0)=0&\Rightarrow c_1=0\\ X(l)=0&\Rightarrow c_2\sin(\lambda l)=0\\ &\Rightarrow \lambda l=n\pi, n=1,2,\cdots\\ &\Rightarrow \lambda=\frac{n\pi}{l}, n=1,2,\cdots\end{aligned}$$

$\therefore X(x)=\sin\frac{n\pi x}{l},\ n=1,2,\cdots$

Now, $\dot{T}=-\lambda^2 T$

$$\begin{aligned}\Rightarrow T(t)&=e^{-\lambda^2 t}\\ &=e^{-\frac{n^2\pi^2}{l^2}t}\end{aligned}$$

$\therefore u(x,t)=e^{-\frac{n^2\pi^2}{l^2}t}\sin\frac{n\pi x}{l},\ n=1,2,\cdots$

The given problem is linear

$\therefore$ The general form of the solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\frac{n^2\pi^2}{l^2}t} \sin\frac{n\pi x}{l},$$

Now, $u(x,0) = sin(\pi x) \quad , \quad 0 < x < l$

$\Rightarrow sin(\pi x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}$

$\Rightarrow A_n = \frac{2}{l}\int_0^l sin(\pi x)\sin\frac{n\pi x}{l}dx$

Let $\int_0^l sin(\pi x)\sin\frac{n\pi x}{l} = \mathrm{I}$

$\Rightarrow \quad I \quad = \quad \left[\frac{-cos(\pi x)}{\pi}\sin\left(\frac{n\pi x}{l}\right)\right]_0^l \quad + \quad \frac{n\pi}{l} \quad \int_0^l \frac{cos(\pi x)}{\pi}\cos\left(\frac{n\pi x}{l}\right)$

$\Rightarrow \quad I \quad = \quad \left[\frac{-cos(\pi x)}{\pi}\sin\left(\frac{n\pi x}{l}\right)\right]_0^l \quad + \quad \frac{n\pi}{l} \quad \left(\quad \left[\frac{sin(\pi x)}{\pi^2}\cos\left(\frac{n\pi x}{l}\right)\right]_0^l \quad + \quad \frac{n\pi}{l} \quad \int_0^l \frac{sin(\pi x)}{\pi^2}\sin\left(\frac{n\pi x}{l}\right)\right)$

$\Rightarrow \quad I \quad = \quad \left[\frac{-cos(\pi x)}{\pi}\sin\left(\frac{n\pi x}{l}\right)\right]_0^l \quad + \quad \frac{n\pi}{l} \quad \left[\frac{sin(\pi x)}{\pi^2}\cos\left(\frac{n\pi x}{l}\right)\right]_0^l \quad + \quad \frac{n^2\pi^2}{l^2} \quad \int_0^l \frac{sin(\pi x)}{\pi^2}\sin\left(\frac{n\pi x}{l}\right)$

$\Rightarrow \quad I \quad = \quad \left[\frac{-cos(\pi x)}{\pi}\sin\left(\frac{n\pi x}{l}\right)\right]_0^l \quad + \quad \frac{n\pi}{l} \quad \left[\frac{sin(\pi x)}{\pi^2}\cos\left(\frac{n\pi x}{l}\right)\right]_0^l \quad + \quad \frac{n^2\pi^2}{l^2\pi^2} \quad I$

$\Rightarrow \quad \left(1 \quad - \quad \frac{n^2}{l^2} \quad\right) \quad I \quad = \quad 0 \quad - \quad 0 \quad + \quad \frac{n}{l \ \pi} \quad [sin(\pi l)] \quad - 0$

$\Rightarrow \quad I \quad = \quad \frac{(-1)^n \ n \ sin(\pi l)}{l \ \pi\left(1 \quad - \quad \frac{n^2}{l^2} \quad\right)}$

$\Rightarrow A_n = \quad \frac{2(-1)^n n \ sin(\pi l)}{\pi(l^2 \quad - \quad n^2 \ )}$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2\,(-1)^n \, n \ sin(\pi l)}{\pi\,(l^2 \quad - \quad n^2 \ )} \quad e^{-\frac{n^2\pi^2}{l^2}t} \sin\frac{n\pi x}{l},$$

(ii)

Stability Using Crank Nicholson Method

$$
\begin{aligned}
u_t &= u_{xx} \\
\Rightarrow \frac{u_j^{n+1} - u_j^n}{k} &= \frac{1}{2}\left[\frac{(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1})}{h^2} + \frac{(u_{j+1}^{n} - 2u_j^{n} + u_{j-1}^{n})}{h^2}\right]
\end{aligned}
\tag{0.0.2}
$$

$$Let \;\; u_j^n \;=\; e^{i\beta jh}\;\lambda^n$$

$Applying \;\; to \;\; (0.0.2) \;\; we \;\; have \;\; ,$

$$e^{i\beta jh}\;\lambda^{n+1} - e^{i\beta jh}\;\lambda^n = \tfrac{r}{2}\left[\tfrac{(e^{i\beta(j+1)h}\lambda^{n+1} - 2e^{i\beta jh}\lambda^{n+1} + e^{i\beta(j-1)h}\lambda^{n+1})}{h^2} + \tfrac{(e^{i\beta(j+1)h}\lambda^{n} - 2e^{i\beta jh}\lambda^{n} + e^{i\beta(j-1)h}\lambda^{n})}{h^2}\right]$$

$$\Rightarrow \lambda - 1 = \tfrac{r}{2}[\lambda+1][e^{i\beta h} + e^{-i\beta h} - 2]$$

$$\Rightarrow \lambda - 1 = \tfrac{r}{2}[\lambda+1][2\cos(\beta h) - 2]$$

$$\Rightarrow \lambda - 1 = -2r[\lambda+1][sin^2(\tfrac{\beta h}{2})]$$

$$\Rightarrow \lambda[1 + r[2sin^2(\tfrac{\beta h}{2})]] = 1 - r[2sin^2(\tfrac{\beta h}{2})]$$

$$\Rightarrow \lambda = \tfrac{1 - r[2sin^2(\frac{\beta h}{2})}{1 + r[2sin^2(\frac{\beta h}{2})}$$

$$\Rightarrow \lambda = \tfrac{1-(+vevalue)}{1+(+vevalue)}$$

$\Rightarrow |\lambda| \ < \ 1 \ \forall \ r$

$\therefore \ Unconditionally \quad Stable$

Consistency Using Crank Nicholson Method

$Let \ U \ be \ the \ exact \ solution \ of \ the \ equation \ u_t \ = u_{xx}$

$Then \ Truncation \ error \ T_j^n \ = F_j^n(U)$

$$\Rightarrow \ T_j^n \ = \frac{U_j^{n+1} - U_j^n}{k} \ - \frac{1}{2}[\frac{(U_{j+1}^{n+1} - 2U_j^{n+1} + u_{j-1}^{n+1})}{h^2} + \frac{(U_{j+1}^n - 2U_j^n + U_{j-1}^n)}{h^2}]$$

$\Rightarrow \ T_j^n \ = \frac{U_j^{n+1} - U_j^n}{k} \ - \frac{1}{2h^2}[(U_{j+1}^{n+1} + U_{j-1}^{n+1}) \ + \ (U_{j-1}^n + (U_{j+1}^n) - 2U_j^{n+1} - 2U_j^n)]$

$$T_j^n \ = \ \frac{1}{2k} \ \left(\ \left(-r[U_{j-1}^{n+1} \ + \ U_{j+1}^{n+1} \ ] + (2+2r)U_j^{n+1}\right) - \left(r[U_{j-1}^n \ + \ U_{j+1}^n \ ] + (2-2r)U_j^n\right)\right)$$

By Taylor's expansion

$$\begin{aligned} U_{j+1}^n &= U((j+1)h, nk) = U(x_j + h, t_n) \\ &= U_j^n + h\left(\frac{\partial U}{\partial x}\right)_{j,n} + \frac{h^2}{2!}\left(\frac{\partial^2 U}{\partial x^2}\right)_{j,n} + \frac{h^3}{6!}\left(\frac{\partial^3 U}{\partial x^3}\right)_{j,n} + \cdots \\ U_{j-1}^n &= U((j-1)h, nk) = U(x_j - h, t_n) \\ &= U_j^n - h\left(\frac{\partial U}{\partial x}\right)_{j,n} + \frac{h^2}{2!}\left(\frac{\partial^2 U}{\partial x^2}\right)_{j,n} - \frac{h^3}{6!}\left(\frac{\partial^3 U}{\partial x^3}\right)_{j,n} + \cdots \end{aligned}$$

$$U_j^{n+1} = U(jh,(n+1)k) = U(x_j, t_n + k)$$
$$= U_j^n + k\left(\frac{\partial U}{\partial t}\right)_{j,n} + \frac{k^2}{2!}\left(\frac{\partial^2 U}{\partial t^2}\right)_{j,n} + \frac{k^3}{3!}\left(\frac{\partial^3 U}{\partial t^3}\right)_{j,n} + \cdots$$
$$U_{j+1}^{n+1} = U((j+1)h,(n+1)k) \quad = \quad U(x_j + h, t_n + k)$$
$$= U_j^n + \left(h\left(\frac{\partial U}{\partial x}\right)_{j,n} + k\left(\frac{\partial U}{\partial t}\right)_{j,n}\right) + \frac{1}{2!}\left(h^2\left(\frac{\partial^2 U}{\partial x^2}\right)_{j,n} + k^2\left(\frac{\partial^2 U}{\partial t^2}\right)_{j,n} + \right.$$
$$\left. 2hk\left(\frac{\partial^2 U}{\partial x \partial t}\right)_{j,n}\right) + \frac{1}{3!}\left(h^3\left(\frac{\partial^3 U}{\partial x^3}\right)_{j,n} + k^3\left(\frac{\partial^3 U}{\partial t^3}\right)_{j,n} + 3h^2k\left(\frac{\partial^2 U}{\partial x^2 \partial t}\right)_{j,n}\right.$$
$$\left. + 3hk^2\left(\frac{\partial^2 U}{\partial x \partial t^2}\right)_{j,n}\right) \cdots$$
$$U_{j-1}^{n+1} = U((j-1)h,(n+1)k)U(x_j - h, t_n + k)$$
$$= U_j^n + \left(-h\left(\frac{\partial U}{\partial x}\right)_{j,n} + k\left(\frac{\partial U}{\partial t}\right)_{j,n}\right) + \frac{1}{2!}\left(h^2\left(\frac{\partial^2 U}{\partial x^2}\right)_{j,n} + k^2\left(\frac{\partial^2 U}{\partial t^2}\right)_{j,n} - \right.$$
$$\left. 2hk\left(\frac{\partial^2 U}{\partial x \partial t}\right)_{j,n}\right) - \frac{1}{3!}\left(h^3\left(\frac{\partial^3 U}{\partial x^3}\right)_{j,n} + k^3\left(\frac{\partial^3 U}{\partial t^3}\right)_{j,n} + 3h^2k\left(\frac{\partial^2 U}{\partial x^2 \partial t}\right)_{j,n}\right.$$
$$\left. - 3hk^2\left(\frac{\partial^2 U}{\partial x \partial t^2}\right)_{j,n}\right) \cdots$$

$$\Rightarrow \frac{U_j^{n+1} - U_j^n}{k} = \left(\frac{\partial U}{\partial t}\right)_{j,n} + \frac{k}{2!}\left(\frac{\partial^2 U}{\partial t^2}\right)_{j,n} + \frac{k^2}{3!}\left(\frac{\partial^3 U}{\partial t^3}\right)_{j,n} + \cdots$$

$$\Rightarrow U_{j-1}^n + U_{j+1}^n \ = \ -2\left[U_j^n \ + \ \frac{h^2}{2!}\left(\frac{\partial^2 U}{\partial x^2}\right)_{j,n} \ + \ \frac{h^4}{4!}\left(\frac{\partial^4 U}{\partial x^4}\right)_{j,n} + \cdots\right]$$

$$\Rightarrow U_{j-1}^{n+1} + U_{j+1}^{n+1} \ = \ -2\left[U_j^n \ + \ k\left(\frac{\partial U}{\partial t}\right)_{j,n} \ + \ \frac{1}{2!}\left(h^2\frac{\partial^2 U}{\partial x^2} \ + \ k^2\frac{\partial^2 U}{\partial t^2}\right)_{j,n} \ + \ \frac{1}{3!}\left(k^3\frac{\partial^3 U}{\partial t^3}\right.\right.$$
$$\left.\left. 3h^2k\frac{\partial^3 U}{\partial x^2 \partial t}\right)_{j,n} \ + O(k) \ + O(h^2)\right]$$

$$-\frac{1}{2h^2}[(U_{j+1}^{n+1} + U_{j-1}^{n+1}) \ + \ (U_{j-1}^n + (U_{j+1}^n) - 2U_j^{n+1} - 2U_j^n)]$$
$$= \frac{-2}{2h^2}\left[\frac{1}{2!}\left(h^2\frac{\partial^2 U}{\partial x^2}\right) \ + \ \frac{1}{2!}\left(h^2\frac{\partial^2 U}{\partial x^2}\right) \ - \ \frac{1}{3!}\left(3h^2k\left(\frac{\partial^3 U}{\partial x^2 \partial t}\right)\right)_{j,n} \ + O(k) \ + O(h^2)\right]$$

$$\Rightarrow T_j^n = \left(\left(\frac{\partial U}{\partial t}\right)_{j,n} - \left(\frac{\partial^2 U}{\partial x^2}\right)_{j,n} + \frac{k}{2!}\left(\frac{\partial^2 U}{\partial t^2}\right)_{j,n} - \frac{1}{2!}\left(k\frac{\partial^3 U}{\partial x^2 \partial t}\right)_{j,n} + O(k) + O(h^2)\right]$$

$$\Rightarrow T_j^n = \left(\left(\frac{\partial U}{\partial t}\right)_{j,n} - \left(\frac{\partial^2 U}{\partial x^2}\right)_{j,n} + \frac{k}{2!}\left(\frac{\partial}{\partial t}\left(\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2}\right)_{j,n}\right) + O(k) + O(h^2)\right]$$

Since U is the EXACT solution of $\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} = 0$

We have

$$T_j^n = \left(\frac{k}{2!}\left(\frac{\partial^2 U}{\partial t^2}\right)_{j,n} + \frac{k^2}{3!}\left(\frac{\partial^3 U}{\partial t^3}\right)_{j,n} + \cdots\right) - \left(\frac{1}{3!}\left(3k\frac{\partial^3 U}{\partial x^2 \partial t}\right)_{j,n} + \frac{2h^2}{4!}\left(\frac{\partial^4 U}{\partial x^4}\right)_{j,n} + \cdots\right)$$

$$\Rightarrow \; T_j^n = O(k) + O(h^2)$$

When $k = rh^2$, we have

$T_j^n = O(k)$ and $T_j^n = O(h^2)$

$$\Rightarrow \quad T_{j,n} \to 0 \quad as \quad h \to 0 \quad and \quad T_{j,n} \to 0 \quad as \quad k \to 0$$

$\therefore$ *System is unconditionally consistent*

(iii)

Numerical solution

$$u_t = u_{xx}$$

$$\Rightarrow \frac{u_j^{n+1} - u_j^n}{k} = \frac{1}{2}[\frac{(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1})}{h^2} + \frac{(u_{j+1}^n - 2u_j^n + u_{j-1}^n)}{h^2}]$$

$$\Rightarrow -ru_{j-1}^{n+1} + (2+2r)u_j^{n+1} - ru_{j+1}^{n+1} = (ru_{j+1}^n + (2-2r)u_j^n + ru_{j-1}^n) \tag{0.0.3}$$

When $x_0 = 0$, $x_f = 1$, $h = 0.04$, $k = 0.01$, $L = 1$ *and* $T = 1$

we have ,

$r = 0.625$, $J = 25$ *and* $N = 100$

Then ((0.0.3)) becomes ,

$$-0.625u_{j-1}^{n+1} + 3.25u_j^{n+1} - 0.625u_{j+1}^{n+1} = (0.625u_{j+1}^n + 0.75u_j^n + 0.625u_{j-1}^n)$$

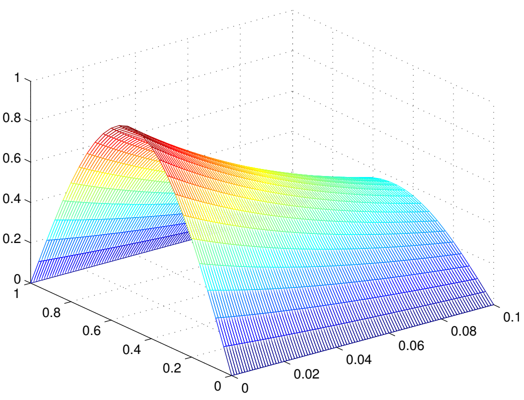
Numerical Solutions

x|t	0	0.001	0.002	.	.	0.099	0.1
0	0	0	0	.	.	0	0
0.04	0.12533	0.124103929	0.128886683	.	.	0.04723629	0.046772983
0.08	0.2486	0.246250666	0.2438359	.	.	0.093727635	0.092808327
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
0.92	0.248689887	0.24625066	0.2438359	.	.	0.093727635	0.092808327
0.96	0.1253333	0.124103929	0.122886683	.	.	0.04723629	0.046772983
1	0	0	0	.	.	0	0

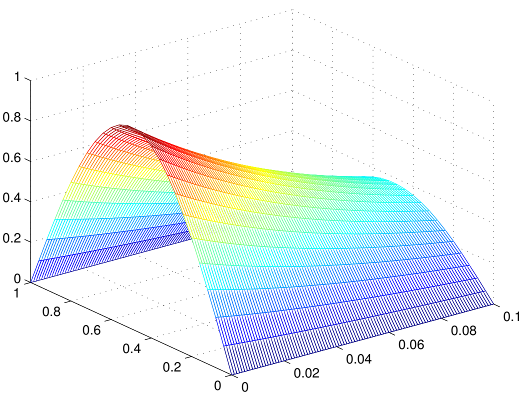
RELATIVE ERROR = 0.0011

Graphs

analytical Solution



Numerical Solution



Question No: 2

Consider the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < l \tag{0.0.4}$$

$IC : u(x, 0) = x^2 - 1, 0 < x < l,$

$BC's : \frac{\partial u}{\partial x}_{(0,t)} = 0, u(l, t) = 0,$

1. Find explicit solution by method of separation of variables.
2. Apply Explicit scheme and check stability and

   consistency.
3. Compute numerical solution for different values of h and k.

ANSWER :

(i)

Assume that $u(x, t) = X(x)T(t)$.

$\Rightarrow u_x = X'T, u_{xx} = X''T, u_t = X\dot{T}$ where $X' = \frac{dX}{dx}, \dot{T} = \frac{dT}{dt}$

Then (0.0.4) becomes

$$X\dot{T} = X''T$$

$$(ie) \quad \frac{\dot{T}}{T} = \frac{X''}{X} = k \ (independent\ of\ t\ and\ x)$$

Suppose $k > 0$.

Then the solution of the equations

$X'' = kX$ is given by

$X(x) = c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x}$

Since $u_x(0,t) = u(l,t) = 0$. we get

$X'(0) = c_1 - c_2 = 0$ and $X(l) = c_1 e^{\sqrt{k}l} + c_2 e^{-\sqrt{k}l} = 0$

$\Rightarrow c_1 = c_2 = 0$.

$\therefore X(x) = 0$.

$\Rightarrow u(x,t) = 0$.

$\therefore$ The given problem has no solution if $f \neq 0$.

Take $k = -\lambda^2$,

Then the solution of the equation

$$X'' = -\lambda^2 X$$

is given by

$X(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x).$

$\Rightarrow X^{'}(x) = c_2 \cos(\lambda x) - c_1 \sin(\lambda x)$

$u_x(0,t) = 0$

$\Rightarrow X'(0) = 0$

$\Rightarrow c_2(1) - c_1(0) = 0$

$\Rightarrow c_2 = 0$

$\Rightarrow \quad X(x) = c_1 \cos(\lambda x)$

$u(l,t) = 0$

$\Rightarrow X(l) = 0$

$\Rightarrow c_2 \cos(\lambda l) = 0$

$\lambda l \; = \; \frac{(2n+1)\pi}{2}$

$\lambda \; = \; \frac{(2n+1)\pi}{2l}$

Now, $\dot{T} = -\lambda^2 T$

$$\Rightarrow T(t) = e^{-\lambda^2 t}$$
$$= e^{-\frac{(2n+1)^2\pi^2}{4l^2}t}$$

$\therefore u(x,t) = e^{-\frac{(2n+1)^2\pi^2 t}{4l^2}} \cos\frac{(2n+1)\pi x}{2l},\ n = 1, 2, \cdots$

Now, $u(x,t) \ = \ \sum_{n=1}^{\infty} A_n \ e^{-\frac{(2n+1)^2\pi^2 t}{4l^2}} \cos\frac{(2n+1)\pi x}{2l}$

Now $u(x,0) = \ x^2 - 1$

$\Rightarrow x^2 - 1 = \ \sum_{n=1}^{\infty} A_n \ \cos\frac{(2n+1)\pi x}{2l}$

$\Rightarrow A_n = \ \frac{2}{l}\int_0^l (x^2 - 1) \ \cos\frac{(2n+1)\pi x}{2l} dx$

(ii)

Stability Using Van Neumann Method

$$u_t \quad = u_{xx}$$
$$\Rightarrow \frac{u_j^{n+1} - u_j^n}{k} \quad = \frac{(u_{j+1}^n - 2u_j^n + u_{j-1}^n)}{h^2} \qquad (0.0.5)$$

$Let \ \ u_j^n \ = \ e^{i\beta jh} \ \lambda^n$

$Applying \ \ to \ \ eq \ \ (3) \ \ we \ \ have \ \ ,$

$$e^{i\beta jh}\ \lambda^{n+1} - e^{i\beta jh}\ \lambda^{n} = \frac{(e^{i\beta(j+1)h}\lambda^{n} - 2e^{i\beta jh}\lambda^{n} + e^{i\beta(j-1)h}\lambda^{n})}{h^2}$$

$$\Rightarrow \lambda - 1 = (r)[e^{i\beta jh} + e^{-i\beta jh} - 2]$$

$$\Rightarrow \lambda - 1 = (r)(2\cos(\frac{\beta h}{2}) - 2)$$

$$\Rightarrow \lambda - 1 = -4r[sin^2(\frac{\beta h}{2})]$$

$$\Rightarrow \lambda = 1 - 4r[sin^2(\frac{\beta h}{2})]$$

$Now, condition\ \ for\ \ stability\ \ is \quad\ \ |\lambda| < 1$

$When \quad |\lambda| < 1 \quad , \quad |1 - 4r[sin^2(\frac{\beta h}{2})]| \leq 1$

$$\Rightarrow \quad -1 \ \ \leq \ \ 1 - 4rsin^2(\frac{\beta h}{2}) \ \ \leq \ \ 1$$

$$\Rightarrow \quad -2 \ \ \leq \ \ -4rsin^2(\frac{\beta h}{2}) \ \ \leq \ \ 0$$

$$\Rightarrow \quad 0 \ \ \leq \ \ 4rsin^2(\frac{\beta h}{2}) \ \ \leq \ \ 2$$

$$\Rightarrow \quad 0 \ \ \leq \ \ rsin^2(\frac{\beta h}{2}) \ \ \leq \ \ \frac{1}{2}$$

$Now\ Maximum\ Value\ of\ sin^2(\frac{\beta h}{2})\ occurs\ when\ \ sin^2(\frac{\beta h}{2}) \ = \ 1\ (ie)\ \frac{\beta h}{2} \ = \ \frac{\pi}{2}$

$At\ \ that\ \ time\ \ , \quad rsin^2(\frac{\beta h}{2}) \ \ \leq \ \ \frac{1}{2} \ \ \Rightarrow \ \ r(1) \ \ \leq \ \ \frac{1}{2} \ \ \Rightarrow \ \ r \ \leq \ \frac{1}{2}$

$\therefore \ \ Maximum\ \ value\ that\ r\ can\ take\ so\ that\ system\ is\ stable\ is\ \frac{1}{2}$

$\therefore \ \ System\ \ is\ \ conditionally\ \ Stable$

Consistency

$$Let\ \ U\ \ be\ \ the\ \ exact\ \ solution\ \ of\ \ the\ \ equation\ \ u_t\ \ = u_{xx}$$

$$Then\ \ Truncation\ \ error\ \ T_j^n\ \ = F_j^n(U)$$

$$\Rightarrow\ \ T_j^n\ \ =\ \ \frac{U_j^{n+1}\ \ -\ \ U_j^n}{k}\ \ -\ \ \frac{U_{j+1}^n\ \ -\ \ 2U_j^n\ \ +\ \ U_{j-1}^n}{h^2} \tag{0.0.6}$$

Now , By Taylor's expansion we have

$$U_{j+1}^n = U_j^n + h(\frac{\partial U}{\partial x}) + \frac{h^2}{2!}(\frac{\partial^2 u}{\partial x^2}) + \frac{h^3}{3!}(\frac{\partial^3 U}{\partial x^3}) + \frac{h^4}{4!}(\frac{\partial^4 U}{\partial x^4}) + \cdots \tag{0.0.7}$$

$$U_{j-1}^n = U_j^n\ \ -\ \ h(\frac{\partial U}{\partial x})\ \ +\ \ \frac{h^2}{2!}(\frac{\partial^2 U}{\partial x^2})\ \ -\ \ \frac{h^3}{3!}(\frac{\partial^3 U}{\partial x^3})\ \ +\ \ \frac{h^4}{4!}(\frac{\partial^4 U}{\partial x^4})\ \ + \cdots \tag{0.0.8}$$

$$U_j^{n+1} = U_j^n\ \ +\ \ k(\frac{\partial U}{\partial t})\ \ +\ \ \frac{k^2}{2!}(\frac{\partial^2 U}{\partial t^2})\ \ -\ \ \frac{k^3}{3!}(\frac{\partial^3 U}{\partial t^3})\ \ +\ \ \frac{k^4}{4!}(\frac{\partial^4 U}{\partial t^4})\ \ + \cdots \tag{0.0.9}$$

$From\ \ (0.0.9)\ \ we\ \ \ have,$

$$\Rightarrow\ \ \frac{U_j^{n+1}\ \ -\ \ U_j^n}{k}\ \ =\ \ \frac{\partial U}{\partial t} + \frac{k}{2!}(\frac{\partial^2 U}{\partial t^2}) - \frac{k^2}{3!}(\frac{\partial^3 U}{\partial t^U}) + \frac{k^3}{4!}(\frac{\partial^4 U}{\partial t^4}) + \cdots \tag{0.0.10}$$

$||^{ly}\ \ From\ \ (0.0.7)\ and\ (0.0.8)\ \ we\ \ \ have,$

$$\frac{U_{j+1}^n\ \ -\ \ 2U_j^n\ \ +\ \ U_{j-1}^n}{h^2}\ \ =\ \ \frac{\partial^2 U}{\partial x^2}\ \ +\ \ \frac{2h^2}{4!}(\frac{\partial^4 U}{\partial x^4})\ \ + \cdots \tag{0.0.11}$$

Substituting results in (8) and (9) into the expression (4) , then gives

$$T_{j,n} = \left(\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2}\right)_{j,n} + \frac{1}{2}k\left(\frac{\partial^2 U}{\partial t^2}\right)_{j,n} - \frac{1}{12}h^2\left(\frac{\partial^4 U}{\partial x^4}\right)_{j,n} + \frac{1}{6}k^2\left(\frac{\partial^3 U}{\partial t^3}\right)_{j,n}$$
$$- \frac{1}{360}h^4\left(\frac{\partial^6 U}{\partial x^6}\right)_{j,n} + \cdots$$

But U is the solution of the differential equation so

$$\left(\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2}\right)_{j,n} = 0$$

Hence

$$T_{j,n} = O(k) + O(h^2).$$

When $k = rh^2, \quad 0 < r \leq \frac{1}{2},\ T_{j,n}$ is $O(k)$ or $O(h^2)$.

$\Rightarrow \quad T_{j,n} \to 0 \quad as \quad h \to \quad 0 \ \ and \ \ T_{j,n} \to 0 \quad as \quad k \to \quad 0$

$\therefore$ *System is unconditionally consistent*

(iii)

Numerical Solution

$\frac{U_j^{n+1} - U_j^n}{k} = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2}$

Taking $r = \frac{k}{h^2}$, we have

$$U_j^{n+1} = rU_{j+1}^n + (1-2r)U_j^n + rU_{j-1}^n \qquad (0.0.12)$$

when t=0 [(ie) j=0] , 5

$$U_0^{n+1} = rU_1^n + (1-2r)U_0^n + rU_{-1}^n \qquad (0.0.13)$$

By initial condition $\frac{\partial u}{\partial x}_{(0,t)} = 0$,we have

$\frac{U_1^n - U_{-1}^n}{2h} = 0 \Rightarrow U_1^n = U_{-1}^n$

Applying this result in eq (11) we have,

$$U_0^{n+1} = 2rU_1^n + (1-2r)U_0^n \qquad (0.0.14)$$

When $x_0 = 0$, $x_f = 0.1$, $y_0 = 0$, $y_f = 1$, $h = 0.001,$ $and\ k = 0.05$

we have ,

$r = 0.4$, $J = 20$ *and* $N = 100$

Then eq (0.0.14) becomes ,

$U_0^{n+1} = 0.8U_1^n + 0.2U_0^n$

eq (0.0.12) becomes

$U_j^{n+1} = 0.4U_{j+1}^n + 0.2U_j^n + 0.4U_{j-1}^n$

Then we get numerical solutions as,

Numerical Solutions

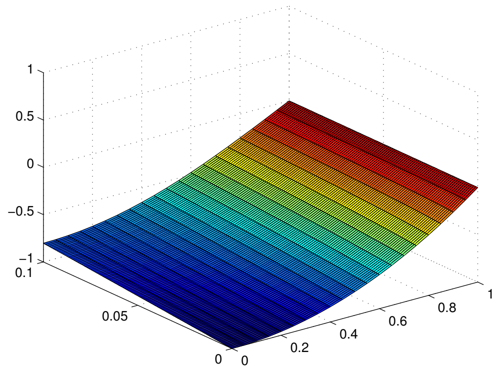
x|t	0	0.001	0.002	.	.	0.99	0.1
0	-1	-0.998	-0.996	.	.	-0.80407	-0.80217
0.05	-0.9975	-0.9955	-0.9935	.	.	-0.80169	-0.79979
0.1	-0.99	-0.988	-0.986	.	.	-0.79457	-0.79268
.	.	.	.	.	.	.	.
0.90	-0.19	-0.188	-0.186	.	.	-0.12835	-0.12799
0.95	-0.0975	-0.0955	-0.0943	.	.	-0.64396	-0.064217
1	0	0	0	.	.	0	0

RELATIVE ERROR $= 1 \times e^{-4}$

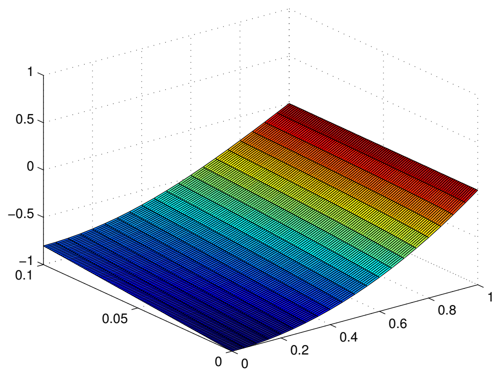
MAXIMUM ITERATION $= 500$

Graphs

analytical Solution



Numerical Solution



Question No: 3

Consider the problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < l \tag{0.0.15}$$

$IC : u(x,0) = \frac{1}{8}\sin \pi x, \frac{\partial u}{\partial t}_{(x,0)} = 0, 0 < x < 1,$

$BC's : u(0,t) = 0, u(1,t) = 0, t > 0$

1. Derive the analytical solution $U = \frac{1}{8} sin\pi x.cos\pi t$ and compare it with numerical solutions at several points

2. Use $explicit \quad finite \quad difference \quad formula$ and a $central \quad - \quad difference \quad approach \quad for \quad derivative \quad condition$ to calculate a solution for x=0(0.1)1 and t=0(0.1)(0.5)

ANSWER :

(i)

GIVEN:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

Assume that $u(x,t) = X(x)T(t)$.

$\Rightarrow u_x = X^{'}T, u_{xx} = X^{''}T, u_t = XT^{'}\ u_{tt} = XT^{''}$ where $X^{'} = \frac{dX}{dx}, T^{'} = \frac{dT}{dt}$

Then (0.0.15) becomes

$$XT^{''} = X^{''}T$$
$$(ie)\quad \frac{T^{''}}{T} = \frac{X^{''}}{X} = k\ (independent\ of\ t\ and\ x)$$

Suppose $k > 0$.

Then the solution of the equations

$X^{''} = kX$ is given by

$X(x) = c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x}$

Since $u(0,t) = u(l,t) = 0$. we get

$x(0) = c_1 + c_2 = 0$ and $X(l) = c_1 e^{\sqrt{k}l} + c_2 e^{-\sqrt{k}l} = 0$

$\Rightarrow c_1 = c_2 = 0$.

$\therefore X(x) = 0$.

$\Rightarrow u(x,t) = 0$.

$\therefore$ The given problem has no solution if $f \neq 0$.

Take $k = -\lambda^2$,

Then the solution of the equation

$X'' = -\lambda^2 X$ is given by

$X(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x)$.

$$\begin{aligned} X(0) = 0 &\Rightarrow c_1 = 0 \\ X(1) = 0 &\Rightarrow c_2 \sin(\lambda) = 0 \\ &\Rightarrow \lambda = n\pi, n = 1, 2, \cdots \\ &\Rightarrow \lambda = \frac{n\pi}{l}, n = 1, 2, \cdots \end{aligned}$$

$\therefore X(x) = c_2 \sin n\pi x,\ \ n = 1, 2, \cdots$

Now, $T'' = -\lambda^2 T$

$\Rightarrow T(t) = c_3 \cos n\pi t + c_4 \sin\ \ n\pi t.$

$\therefore u(x,t) = c_2 \sin n\pi x[c_3 \cos n\pi t + c_4 \sin n\pi t],\ \ n = 1, 2, \cdots$

The given problem is linear

$\therefore$ The general form of the solution is

$$u(x,t) = \sum_{n=1}^{\infty} [A_n \sin n\pi x \cos n\pi t + B_n \sin n\pi x \sin n\pi t], \qquad (0.0.16)$$

$$\Rightarrow u_t(x,t) = \sum_{n=1}^{\infty} [A_n \sin n\pi x(-\sin n\pi t) + B_n \sin n\pi x \cos n\pi t][n\pi], \qquad (0.0.17)$$

Now, Applying $u_t(x,0) = 0$ in (0.0.17) , we get

$\sum_{n=1}^{\infty} [A_n \sin n\pi x(0) + B_n \sin n\pi x(1)][n\pi] \quad = \quad 0$

$\Rightarrow B_n \sin n\pi x[n\pi] \quad = \quad 0 \quad \forall \;\; n$

$\Rightarrow B_n \quad = \quad 0 \quad \forall \;\; n$

Applying this in eq (11), we get

$$u(x,t) = \sum_{n=1}^{\infty}[A_n \sin n\pi x(\cos n\pi t)] \tag{0.0.18}$$

Applying boundary condition $u(x,0) = \frac{1}{8}\sin \pi x$ in eq (13) we get,

$\sum_{n=1}^{\infty}[A_n \sin n\pi x] \quad = \quad \frac{1}{8}\sin \pi x$

$$\Rightarrow A_n = \frac{2}{1}\int_0^1 \frac{1}{8}\sin(\pi x)\sin(n\pi x)\; dx \tag{0.0.19}$$

Let $\int_0^1 sin(\pi x)\sin n\pi x = \mathrm{I}$

$\Rightarrow \quad I \quad = \quad \left[\frac{-cos(\pi x)}{\pi}\sin\left(n\pi x\right)\right]_0^1 \quad + \quad n\pi \quad \int_0^1 \frac{cos(\pi x)}{\pi}\cos\left(n\pi x\right)$

$\Rightarrow \quad I \quad = \quad \left[\frac{-cos(\pi x)}{\pi}\sin\left(n\pi x\right)\right]_0^1 \quad + \quad n\pi \quad \left(\quad \left[\frac{sin(\pi x)}{\pi^2}\cos\left(n\pi x\right)\right]_0^1 \quad + \quad n\pi \quad \int_0^1 \frac{sin(\pi x)}{\pi^2}\sin\left(n\pi x\right)\right)$

$\Rightarrow \quad I \quad = \quad \left[\frac{-cos(\pi x)}{\pi}\sin\left(n\pi x\right)\right]_0^1 \quad + \quad n\pi \quad \left[\frac{sin(\pi x)}{\pi^2}\cos\left(n\pi x\right)\right]_0^1 \quad + n^2\pi^2 \quad \int_0^1 \frac{sin(\pi x)}{\pi^2}\sin\left(n\pi x\right)$

$\Rightarrow \quad I \quad = \quad \left[\frac{-cos(\pi x)}{\pi}\sin\left(n\pi x\right)\right]_0^1 \quad + \quad n\pi \quad \left[\frac{sin(\pi x)}{\pi^2}\cos\left(n\pi x\right)\right]_0^1 \quad + \quad \frac{n^2\pi^2}{\pi^2} \quad I$

$\Rightarrow \quad (1 \quad - \quad n^2 \;) \;\; I \quad = \;\; 0 \quad - \quad 0 \;\; + \quad \frac{n}{\pi} \quad [sin(\pi)] \quad - 0$

$\Rightarrow \quad I \quad = \quad \frac{(-1)^n \;\; n \;\; sin(\pi)}{l \;\; \pi(1 \quad - \quad n^2 \;\;)}$

$\Rightarrow A_n = \quad \frac{2(-1)^n n \;\; sin(\pi l)}{\pi(1 \quad - \quad n^2 \;\;)}$

$\Rightarrow A_n = 0 \;\forall\;$ *values of* n *except* $n \;=\; 1 \;(\frac{0}{0}$ *form*)

When n=1 , we have

$A_1 = \frac{2}{1}\int_0^1 \frac{1}{8}\sin(\pi x)\;.\;\sin(\pi x)\;dx$

$\Rightarrow\; A_1 = \frac{1}{4}\int_0^l \sin^2\pi x\;dx$

$\Rightarrow\; A_1 = \frac{1}{4}\int_0^l (\frac{1-\cos 2\pi x}{2})\;dx$

$\Rightarrow\; A_1 = \frac{1}{8}\;[x - \frac{\sin 2\pi x}{2\pi}]_0^1$

$\Rightarrow\; A_1 = \frac{1}{8}\;[1-0-0+0]$

$\Rightarrow\; A_1 = \frac{1}{8}$

Applying this result in eq (15) we get

$u(x,t) = \;\frac{1}{8}\sin\pi x(\cos\pi t)$

(ii)

Numerical Solution

Given that

$$\begin{aligned} u_{tt} &= u_{xx} \\ \Rightarrow \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{k^2} &= \frac{(u_{j+1}^n - 2u_j^n + u_{j-1}^n)}{h^2} \end{aligned} \tag{0.0.20}$$

Taking $r = k/h$ we have,

$$u_j^{n+1} = r^2 u_{j-1}^n + 2(1-r^2)u_j^n + r^2 u_{j+1}^n - u_j^{n-1} \tag{0.0.21}$$

when n=0 we have,

$$u_j^1 = r^2 u_{j-1}^0 + 2(1-r^2)u_j^0 + r^2 u_{j+1}^0 - u_j^{-1} \tag{0.0.22}$$

Now, $\frac{\partial u}{\partial x}_{(0,t)} = 0$

$\Rightarrow \frac{u_j^1 - u_j^{-1}}{2k} = 0$

$\Rightarrow u_j^1 = u_j^{-1}$

Applying this result in eq (0.0.22) we have

$$u_j^1 = \frac{1}{2}[\ r^2 u_{j-1}^0 + 2(1-r^2)u_j^0 + r^2 u_{j+1}^0\ ] \tag{0.0.23}$$

When $x_0 = 0$, $x_f = 1$, $h = 0.04$, $k = 0.001$, $L = 1$ *and* $T = 0.1$

we have , $r = 6.25 \times e^{-04}$, $J = 25$ *and* $N = 100$

Based on the eq (0.0.23) and eq (0.0.21) ,

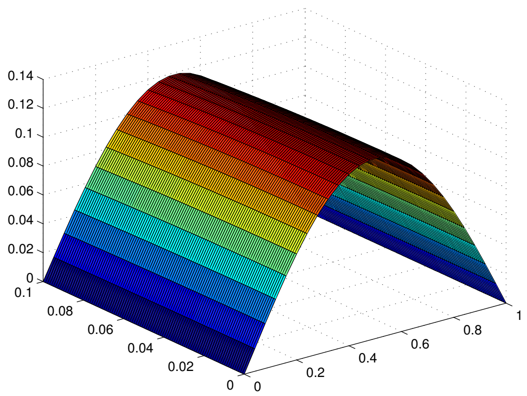
we have the following numerical solutions

x ⌒	0	0.001	0.002	.	.	0.099	0.1
0	0	0	0	.	.	0	0
0.04	0.0156666542	0.0156666541	0.015666654	.	.	0.0156661812	0.0156661716
0.08	0.0310862359	0.0310862358	0.0310862355	.	.	0.0310852974	0.0310852784
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
0.88	0.0460155691	0.0460155689	0.0460155685	.	.	0.0460141799	0.0460141517
0.92	0.0310862359	0.0310862358	0.0310862355	.	.	0.0310852974	0.0310852784
0.96	0.0156666542	0.0156666541	0.015666654	.	.	0.0156661812	0.0156661716
1	0	0	0	.	.	0	0

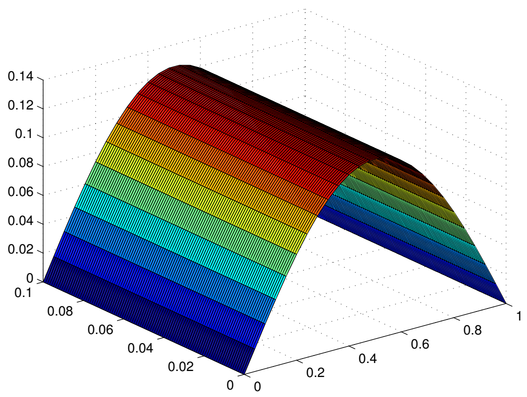
RELATIVE ERROR $= 4.931\ \times e^{-0.4}$

Graphs

analytical Solution



Numerical Solution



Question No: 4

Consider the problem

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \quad , \qquad 0 < x < l$$

$IC's : u(x,0) = \begin{Bmatrix} x \ , 0 \ < x < \frac{l}{2} \\ l - x \ , \frac{l}{2} \ < x < l \end{Bmatrix} \ , \ \frac{\partial u}{\partial t}(x,0) = 0 \ , \quad 0 \ < \ x \ < \ l$

$BC's : u(0,t) = 0, u(l,t) = 0, t > 0$

1. Find explicit solution by method of separation of variables.
2. Apply Crank- Nicholson method and check stability and

   consistency.

3. Compute numerical solution for different values of t by taking h=k.

ANSWER :

(i)

GIVEN:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

Assume that $u(x,t) = X(x)T(t)$.

$\Rightarrow u_x = X^{'}T, u_{xx} = X^{''}T, u_t = XT^{'} \; u_{tt} = XT^{''}$ where $X^{'} = \frac{dX}{dx}, T^{'} = \frac{dT}{dt}$

Then eq (10) becomes

$$XT^{''} = a^2 X^{''} T$$
$$(ie) \quad \frac{T^{''}}{a^2 T} = \frac{X^{''}}{X} = k \; (independent \; of \; t \; and \; x)$$

Suppose $k > 0$.

Then the solution of the equations

$X^{''} = kX$ is given by

$X(x) = c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x}$

Since $u(0,t) = u(l,t) = 0$. we get

$x(0) = c_1 + c_2 = 0$ and $X(l) = c_1 e^{\sqrt{k}l} + c_2 e^{-\sqrt{k}l} = 0$

$\Rightarrow c_1 = c_2 = 0.$

$\therefore X(x) = 0.$

$\Rightarrow u(x,t) = 0.$

$\therefore$ The given problem has no solution if $f \neq 0$.

Take $k = -\lambda^2$,

Then the solution of the equation

$X'' = -\lambda^2 X$ is given by

$X(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x)$.

$$\begin{aligned} X(0) = 0 &\Rightarrow c_1 = 0 \\ X(l) = 0 &\Rightarrow c_2 \sin(\lambda l) = 0 \\ &\Rightarrow \lambda l = n\pi, n = 1, 2, \cdots \\ &\Rightarrow \lambda = \frac{n\pi}{l}, n = 1, 2, \cdots \end{aligned}$$

$\therefore X(x) = c_2 \sin \frac{n\pi x}{l},\ n = 1, 2, \cdots$

Now, $T'' = -a^2 \lambda^2 T$

$\Rightarrow T(t) = c_3 \cos \frac{n\pi a t}{l} + c_4 \sin \frac{n\pi a t}{l}$.

$\therefore u(x,t) = c_2 \sin \frac{n\pi x}{l}[c_3 \cos \frac{n\pi a t}{l} + c_4 \sin \frac{n\pi a t}{l}],\ n = 1, 2, \cdots$

The given problem is linear

$\therefore$ The general form of the solution is

$$u(x,t) = \sum_{n=1}^{\infty}[A_n \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l} + B_n \sin \frac{n\pi x}{l} \sin \frac{n\pi a t}{l}],$$

$$\Rightarrow u_t(x,t) = \sum_{n=1}^{\infty}[A_n \sin \frac{n\pi x}{l}(-\sin \frac{n\pi a t}{l}) + B_n \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l}][\frac{n\pi a}{l}],$$

Now, Applying $u_t(x,0)=0$ in eq (12), we get

$\sum_{n=1}^{\infty}[A_n \sin\frac{n\pi x}{l}(0)+B_n \sin\frac{n\pi x}{l}(1)][\frac{n\pi}{l}] \quad = \quad 0$

$\Rightarrow B_n \sin\frac{n\pi x}{l}[\frac{n\pi a}{l}] \quad = \quad 0 \quad \forall \;\; n$

$\Rightarrow B_n \quad = \quad 0 \quad \forall \;\; n$

Applying this in eq (11), we get

$$u(x,t)=\sum_{n=1}^{\infty}[A_n \sin\frac{n\pi x}{l}(\cos\frac{n\pi at}{l})]$$

Applying boundary condition $u(x,0)=f(x)$ in eq (13) we get,

$\sum_{n=1}^{\infty}[A_n \sin\frac{n\pi x}{l}] \quad = \quad f(x)$

$$\Rightarrow A_n = \left\{\begin{array}{l} \frac{2}{l}\ \int_0^{\frac{l}{2}} x\sin(\frac{n\pi x}{l})dx, \quad 0 < x < \frac{l}{2} \\ \frac{2}{l}\ \int_{\frac{l}{2}}^{l}(l-x)\sin(\frac{n\pi x}{l})dx, \quad \frac{l}{2} < x < l \end{array}\right\}$$

$$\begin{aligned} \frac{2}{l}\ \int_0^{\frac{l}{2}} x\sin(\frac{n\pi x}{l})dx \quad &= \quad \frac{2}{l}\left[x\left(\frac{-\cos(\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) \quad + \quad \left(\frac{\sin(\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_0^{\frac{l}{2}} \\ &= \quad \left(\frac{2\ l\ \sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}\right) \end{aligned}$$

$$\begin{aligned} \frac{2}{l}\ \int_{\frac{l}{2}}^{l}(l-x)\sin(\frac{n\pi x}{l})dx \quad &= \quad \frac{2}{l}\left[(l-x)\left(\frac{-\cos(\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) \quad - \quad \left(\frac{\sin(\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)\right]_{\frac{l}{2}}^{l} \\ &= \quad \left(\frac{2\ l\ \sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}\right) \end{aligned}$$

$$\Rightarrow A_n = \left\{ \frac{2\ l\ \sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2},\quad 0\ <\ x\ <\ l \right\}$$

$\Rightarrow$

$$\Rightarrow u\left(x,t\right) = \sum_{n=1}^{\infty} \frac{2\ l\ \sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2} \sin\frac{n\pi x}{l}(\cos\frac{n\pi at}{l}),\quad 0\ <\ x\ <\ l$$

(ii)

Stability Using Crank Nicholson Method

Given that

$$u_{tt} \quad = a^2 u_{xx}$$

$$\Rightarrow \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{k^2} \quad = \frac{a^2}{2h^2}\left[\left(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}\right) \quad + \quad \left(u_{j+1}^{n-1} - 2u_j^{n-1} + u_{j-1}^{n-1}\right)\right] \tag{0.0.24}$$

Let $u_j^n \ = \ e^{i\beta jh}\ \lambda^n \quad , \quad \frac{ak}{h} \ = \ \mu$

Applying to (0.0.24) *we have* ,

$$e^{i\beta jh}\ \lambda^{n+1} - 2e^{i\beta jh}\ \lambda^n + e^{i\beta jh}\ \lambda^{n-1} \quad = \quad \frac{\mu^2}{2}\left[\left(e^{i\beta(j+1)h}\lambda^{n+1} - 2e^{i\beta jh}\lambda^{n+1} + e^{i\beta(j-1)h}\lambda^{n+1}\right)\right.$$
$$\left. + \quad \left(e^{i\beta(j+1)h}\lambda^{n-1} - 2e^{i\beta jh}\lambda^{n-1} + e^{i\beta(j-1)h}\lambda^{n-1}\right)\right]$$

Dividing by $e^{i\beta jh}\ \lambda^n$, we have

$$\lambda\ -\ 2\ +\ \frac{1}{\lambda}\ =\ \frac{\mu^2}{2}\left[\lambda\left(e^{i\beta jh}+e^{-i\beta jh}-2\right)\quad+\quad\frac{\left(e^{i\beta jh}+e^{-i\beta jh}-2\right)}{\lambda}\right]$$

$$\Rightarrow\ \lambda\ -\ 2\ +\ \frac{1}{\lambda}\ =\ \mu^2\left[\lambda\left(\cos(\beta h)-1\right)\quad+\quad\frac{(\cos(\beta h)-1)}{\lambda}\right]$$

$$\Rightarrow\ \lambda^2\ -\ 2\lambda\ +\ 1\ =\ \lambda^2\left(-2\mu^2 sin^2(\tfrac{\beta h}{2})\right)\quad+\quad\left(-2\mu^2 sin^2(\tfrac{\beta h}{2})\right)$$

$$\Rightarrow\ \lambda^2\left(1+2\mu^2 sin^2(\tfrac{\beta h}{2})\right)\ -\ 2\lambda\ +\ \left(1+2\mu^2 sin^2(\tfrac{\beta h}{2})\right)$$

$$\Rightarrow\ \lambda\ =\ \frac{-(-2)\ \pm\ \sqrt{(-2)^2-4(1\ +\ 2\mu^2 sin^2\frac{\beta h}{2})^2}}{2(1\ +\ 2\mu^2 sin^2\frac{\beta h}{2}))}$$

$$\Rightarrow\ \lambda\ =\ \frac{1\ \pm\ 2i\sqrt{(1\ +\ \mu^2 sin^2\frac{\beta h}{2})(\ \mu^2 sin^2\frac{\beta h}{2})}}{2(1\ +\ 2\mu^2 sin^2\frac{\beta h}{2}))}$$

$$\Rightarrow\ \ |\lambda|^2\ =\ \ 1$$

Hence System is Unconditionally Stable.

Consistency Using Crank Nicholson Method

Let U be the exact solution of the equation $u_{tt} = u_{xx}$

Then Truncation error $T_j^n = F_j^n(U)$

$$\Rightarrow T_j^n = \frac{U_j^{n+1} - 2U_j^n + U_j^{n-1}}{k^2} - \frac{a^2}{2h^2}\left[\left(U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}\right) + \left(U_{j+1}^{n-1} - 2U_j^{n-1} + U_{j-1}^{n-1}\right)\right]$$

$$\Rightarrow T_j^n = \frac{U_j^{n+1} - 2U_j^n + U_j^{n-1}}{k^2} - a^2\left[\frac{\left(U_j^{n+1} + U_{j-1}^{n+1}\right) + \left(U_{j+1}^{n-1} + U_{j-1}^{n-1}\right) - 2U_j^{n+1} - 2U_j^{n-1}}{2h^2}\right]$$

By Taylor's expansion

$$\begin{aligned} U_j^{n+1} &= U(jh,(n+1)k) = U(x_j, t_n + k) \\ &= U_j^n + k\left(\frac{\partial U}{\partial t}\right)_{j,n} + \frac{k^2}{2!}\left(\frac{\partial^2 U}{\partial t^2}\right)_{j,n} + \frac{k^3}{3!}\left(\frac{\partial^3 U}{\partial t^3}\right)_{j,n} + \cdots \\ U_j^{n-1} &= U(jh,(n-1)k) = U(x_j, t_n - k) \\ &= U_j^n - k\left(\frac{\partial U}{\partial t}\right)_{j,n} + \frac{k^2}{2!}\left(\frac{\partial^2 U}{\partial t^2}\right)_{j,n} - \frac{k^3}{3!}\left(\frac{\partial^3 U}{\partial t^3}\right)_{j,n} + \cdots \end{aligned}$$

$$\Rightarrow \left(U_j^{n+1} + U_j^{n-1}\right) = 2\left(U_j^n + \frac{k^2}{2!}\left(\frac{\partial^2 U}{\partial t^2}\right)_{j,n} + \frac{k^4}{4!}\left(\frac{\partial^4 U}{\partial t^4}\right)_{j,n} + \cdots\right)$$

$$\Rightarrow \frac{U_j^{n+1} - 2U_j^n + U_j^{n-1}}{k^2} = \left(\frac{\partial^2 U}{\partial t^2}\right)_{j,n} + O(k^2)$$

By Taylor's expansion

$$\begin{aligned} U_{j+1}^{n+1} &= U((j+1)h,(n+1)k) = U(x_j + h, t_n + k) \\ &= U_j^n + \left(h\left(\frac{\partial U}{\partial x}\right)_{j,n} + k\left(\frac{\partial U}{\partial t}\right)_{j,n}\right) + \frac{1}{2!}\left(h^2\left(\frac{\partial^2 U}{\partial x^2}\right)_{j,n} + k^2\left(\frac{\partial^2 U}{\partial t^2}\right)_{j,n} + \right. \end{aligned}$$

$$2hk\left(\frac{\partial^2 U}{\partial x\partial t}\right)_{j,n}\Bigg) + \frac{1}{3!}\Bigg(h^3\left(\frac{\partial^3 U}{\partial x^3}\right)_{j,n} + k^3\left(\frac{\partial^3 U}{\partial t^3}\right)_{j,n} + 3h^2k\left(\frac{\partial^2 U}{\partial x^2\partial t}\right)_{j,n}$$

$$+ 3hk^2\left(\frac{\partial^2 U}{\partial x\partial t^2}\right)_{j,n}\Bigg)\cdots$$

$$U_{j-1}^{n+1} = U((j-1)h,(n+1)k) \quad = \quad U(x_j - h, t_n + k)$$

$$= U_j^n + \left(-h\left(\frac{\partial U}{\partial x}\right)_{j,n} + k\left(\frac{\partial U}{\partial t}\right)_{j,n}\right) + \frac{1}{2!}\Bigg(h^2\left(\frac{\partial^2 U}{\partial x^2}\right)_{j,n} + k^2\left(\frac{\partial^2 U}{\partial t^2}\right)_{j,n} -$$

$$2hk\left(\frac{\partial^2 U}{\partial x\partial t}\right)_{j,n}\Bigg) - \frac{1}{3!}\Bigg(h^3\left(\frac{\partial^3 U}{\partial x^3}\right)_{j,n} + k^3\left(\frac{\partial^3 U}{\partial t^3}\right)_{j,n} + 3h^2k\left(\frac{\partial^2 U}{\partial x^2\partial t}\right)_{j,n}$$

$$- 3hk^2\left(\frac{\partial^2 U}{\partial x\partial t^2}\right)_{j,n}\Bigg)\cdots$$

$$\Rightarrow \left(U_{j+1}^{n+1} \; + \; U_{j-1}^{n+1}\right) \; = 2\Bigg(U_j^n \quad + \quad k\left(\frac{\partial U}{\partial t}\right)_{j,n} \quad + \quad + \frac{1}{2!}\Bigg(h^2\left(\frac{\partial^2 U}{\partial x^2}\right)_{j,n} + k^2\left(\frac{\partial^2 U}{\partial t^2}\right)_{j,n}\Bigg) \quad +$$

$$\frac{1}{3!}\Bigg(k^3\left(\frac{\partial^3 U}{\partial t^3}\right)_{j,n} + 3h^2k\left(\frac{\partial^2 U}{\partial x^2\partial t}\right)_{j,n}\Bigg) + \cdots\Bigg)$$

By Taylor's expansion

$$U_{j+1}^{n-1} = U((j+1)h,(n-1)k) \quad = \quad U(x_j + h, t_n - k)$$

$$= U_j^n + \left(h\left(\frac{\partial U}{\partial x}\right)_{j,n} - k\left(\frac{\partial U}{\partial t}\right)_{j,n}\right) + \frac{1}{2!}\Bigg(h^2\left(\frac{\partial^2 U}{\partial x^2}\right)_{j,n} + k^2\left(\frac{\partial^2 U}{\partial t^2}\right)_{j,n} -$$

$$2hk\left(\frac{\partial^2 U}{\partial x\partial t}\right)_{j,n}\Bigg) + \frac{1}{3!}\Bigg(h^3\left(\frac{\partial^3 U}{\partial x^3}\right)_{j,n} - k^3\left(\frac{\partial^3 U}{\partial t^3}\right)_{j,n} - 3h^2k\left(\frac{\partial^2 U}{\partial x^2\partial t}\right)_{j,n}$$

$$+ 3hk^2\left(\frac{\partial^2 U}{\partial x\partial t^2}\right)_{j,n}\Bigg)\cdots$$

$$U_{j-1}^{n-1} = U((j-1)h,(n-1)k) \; = \; U(x_j - h, t_n - k)$$

$$= U_j^n + \left(-h\left(\frac{\partial U}{\partial x}\right)_{j,n} - k\left(\frac{\partial U}{\partial t}\right)_{j,n}\right) + \frac{1}{2!}\Bigg(h^2\left(\frac{\partial^2 U}{\partial x^2}\right)_{j,n} + k^2\left(\frac{\partial^2 U}{\partial t^2}\right)_{j,n} -$$

$$2hk\left(\frac{\partial^2 U}{\partial x\partial t}\right)_{j,n}\Bigg) - \frac{1}{3!}\Bigg(h^3\left(\frac{\partial^3 U}{\partial x^3}\right)_{j,n} - k^3\left(\frac{\partial^3 U}{\partial t^3}\right)_{j,n} - 3h^2k\left(\frac{\partial^2 U}{\partial x^2\partial t}\right)_{j,n}$$

$$- 3hk^2\left(\frac{\partial^2 U}{\partial x\partial t^2}\right)_{j,n}\Bigg)\cdots$$

$$\Rightarrow \left(U_{j+1}^{n-1} + U_{j-1}^{n-1}\right) = 2\left(U_j^n - k\left(\frac{\partial U}{\partial t}\right)_{j,n} + \frac{1}{2!}\left(h^2\left(\frac{\partial^2 U}{\partial x^2}\right)_{j,n} + k^2\left(\frac{\partial^2 U}{\partial t^2}\right)_{j,n}\right) - \frac{1}{3!}\left(k^3\left(\frac{\partial^3 U}{\partial t^3}\right)_{j,n} + 3h^2 k\left(\frac{\partial^2 U}{\partial x^2 \partial t}\right)_{j,n}\right) + \cdots\right)$$

$$\left(U_j^{n+1} + U_{j-1}^{n+1}\right) + \left(U_{j+1}^{n-1} + U_{j-1}^{n-1}\right) - 2U_j^{n+1} - 2U_j^{n-1} = \frac{2}{2!}\left(2h^2\frac{\partial^2 U}{\partial x^2}\right) + O(h^4)$$

$$-a^2\frac{\left(U_j^{n+1}+U_{j-1}^{n+1}\right)+\left(U_{j+1}^{n-1}+U_{j-1}^{n-1}\right)-2U_j^{n+1}-2U_j^{n-1}}{2h^2} = -a^2\left(\frac{\partial^2 U}{\partial x^2}\right)_{j,n} + O(h^2)$$

$$\Rightarrow T_j^n = \frac{U_j^{n+1}-2U_j^n+U_j^{n-1}}{k^2} - a^2\frac{\left(U_j^{n+1}+U_{j-1}^{n+1}\right)+\left(U_{j+1}^{n-1}+U_{j-1}^{n-1}\right)-2U_j^{n+1}-2U_j^{n-1}}{2h^2}$$
$$= \left(\left(\frac{\partial^2 U}{\partial t^2}\right)_{j,n} - a^2\left(\frac{\partial^2 U}{\partial x^2}\right)_{j,n}\right) + O(k^2) + O(h^2)$$

Since U is the exact solution of $\frac{\partial^2 U}{\partial t^2} - a^2\frac{\partial^2 U}{\partial x^2} = 0$

$T_j^n = O(k^2) + O(h^2)$

$\Rightarrow T_{j,n} \to 0 \quad as \quad h \to 0 \ and \ T_{j,n} \to 0 \quad as \quad k \to 0$

$\therefore$ *System is unconditionally consistent*

(iii)

Numerical Solution

Question No: 5

Consider the problem ,$\bigtriangledown u = 0$ on 0 $<$ x $<$ a ,0 $<$ y $<$ b

$u(x, 0) \;=\; x \;,\; 0 \;<\; x \;<\; a$

$u(0, y) \;=\; 0 \;,\; 0 \;<\; y \;<\; b$

$u(x, b) \;=\; x^2 \;,\; 0 \;<\; x \;<\; a$

$u(a, y) \;=\; 0 \;,\; 0 \;<\; y \;<\; b$

1. Find explicit solution by method of Seperation of Variables

2. Discretize and Find Numerical Solution

ANSWER :

(i)

GIVEN:

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} \;=\; 0 \tag{0.0.25}$$

Assume that $u(x, t) = X(x)Y(y)$.

$\Rightarrow\; u_x \;=\; X'Y, u_{xx} \;=\; X''Y, u_y \;=\; XY' \;\; u_{yy} \;=\; XY''$ where $X' \;=$

$\frac{dX}{dx}, Y' = \frac{dY}{dy}$

Then eq (10) becomes

$$\begin{aligned} XY'' + X''Y &= 0 \\ (ie) \qquad \frac{-Y''}{Y} &= \frac{X''}{X} = k \ (independent\ of\ x\ and\ y) \end{aligned}$$

Suppose $k > 0$.

Then the solution of the equations

$X'' = kX$ is given by

$X(x) = c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x}$

Since $u(o, y) = u(a, y) = 0$. we get

$x(0) = c_1 + c_2 = 0$ and $X(a) = c_1 e^{\sqrt{k}a} + c_2 e^{-\sqrt{k}a} = 0$

$\Rightarrow c_1 = c_2 = 0.$

$\therefore X(x) = 0.$

$\Rightarrow u(x, y) = 0.$

$\therefore$ The given problem has no solution if $f \neq 0$.

Take $k = -\lambda^2$,

Then the solution of the equation

$X'' = -\lambda^2 X$ is given by

$X(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x).$

$$\begin{aligned} X(0) = 0 &\Rightarrow c_1 = 0 \\ X(l) = 0 &\Rightarrow c_2 \sin(\lambda a) = 0 \\ &\Rightarrow \lambda a = n\pi, n = 1, 2, \cdots \\ &\Rightarrow \lambda = \frac{n\pi}{a}, n = 1, 2, \cdots \end{aligned}$$

$\therefore X(x) = c_2 \sin \frac{n\pi x}{a},\ n = 1, 2, \cdots$

Now, $Y'' = --\lambda^2 Y$

$\Rightarrow Y(y) = c_3 e^{\frac{n\pi}{a}} + c_4 e^{-\frac{n\pi t}{a}}$

$\therefore u(x, y) = c_2 \sin \frac{n\pi x}{a} [c_3 e^{\frac{n\pi y}{a}} + c_4 e^{-\frac{n\pi y}{a}}],\ n = 1, 2, \cdots$

The given problem is linear

$\therefore$ The general form of the solution is

$$u(x, y) = \sum_{n=1}^{\infty} [A_n \sin \frac{n\pi x}{a} e^{\frac{n\pi y}{a}} + B_n \sin \frac{n\pi x}{a} e^{-\frac{n\pi y}{a}}],$$

Now $u(x, 0) = x \Rightarrow \sum_{n=1}^{\infty} [A_n + B_n] \sin \frac{n\pi x}{a} = x$

$\Rightarrow A_n + B_n = \frac{2}{a} \int_0^a x \sin(\frac{n\pi x}{a}) dx$

Now $u(x, b) = x^2 \Rightarrow \sum_{n=1}^{\infty} [A_n e^{\frac{n\pi b}{a}} + B_n e^{-\frac{n\pi b}{a}}] \sin \frac{n\pi b}{a} = x^2$

$\Rightarrow A_n e^{\frac{n\pi b}{a}} + B_n e^{-\frac{n\pi b}{a}} = \frac{2}{a} \int_0^a x^2 \sin(\frac{n\pi x}{a}) dx$

A_n and B_n can be found using Fourier Integrals

(ii)

Numerical solution

GIVEN:

$\bigtriangledown u = 0$, $0 <$ x $<$ a

$ie \quad \frac{\partial^2 u}{\partial y^2} \quad + \quad \frac{\partial^2 u}{\partial x^2} \quad = \quad 0$

$$\Rightarrow \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{k^2} \quad + \quad \frac{\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)}{h^2} \quad = \quad 0 \quad (0.0.26)$$

When $x_0 = 0 \; , x_f = 1 \; , y_0 = 0 \; , y_f = 1 \; , h = 0.05, \quad and \; k = 0.05$

we have ,

$r = 20 \; , \; J = 20 \; and \; N = 20$

Since k=h=0.05, eq (0.0.29) becomes,

$u_j^{n+1} - 4u_j^n + u_j^{n-1} \quad + \quad u_{j+1}^n \; + u_{j-1}^n \quad = \quad 0$

then we get numerical solutions as,

Numerical Solutions

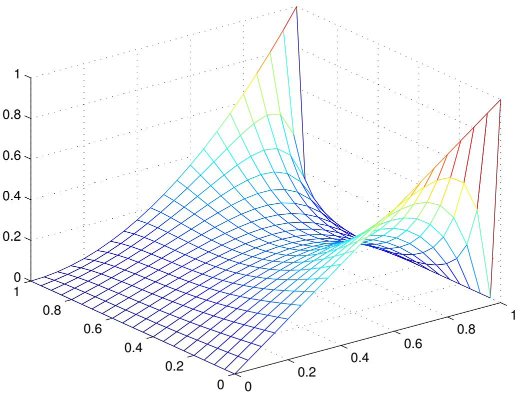
x|t	0	0.05	0.1	.	.	0.95	1
0	0	0.05	0.1	.	.	0.95	1
0.05	0	0.0467712745	0.0934777359	.	.	0.452184948	0
0.1	0	0.0435997148	0.0870728323	.	.	0.2567498404	0
.	.	.	.	.	.	.	.
0.90	0	0.0131548311	0.0276350163	.	.	0.2263025108	0
0.95	0	0.0090347859	0.0204750973	.	.	0.4144498663	0
1	0	0.0025	0.01	.	.	0.9025	1

RELATIVE ERROR $= 1 \times e^{-4}$

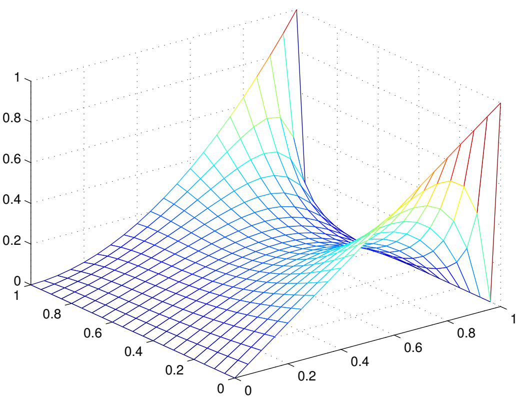
MAXIMUM ITERATION $= 500$

Graphs

analytical Solution



Numerical Solution



Question No: 6

Consider the problem ,$\bigtriangledown u = -1$ on 0 < x < a ,0 < y < b

$u(x,0) \ = \ 0 \ , \ 0 \ < \ x \ < \ a$

$u(0,y) \ = \ 0 \ , \ 0 \ < \ y \ < \ b$

$u(x,b) \ = \ 0 \ , \ 0 \ < \ x \ < \ a$

$u(a,y) \ = \ 0 \ , \ 0 \ < \ y \ < \ b$

1. Find explicit solution by method of Seperation of Variables

2. Discretize and Find Numerical Solution

ANSWER :

(i)

GIVEN:

$\bigtriangledown u = -1$

$ie \quad \frac{\partial^2 u}{\partial x^2} \quad + \quad \frac{\partial^2 u}{\partial y^2} \quad = \quad -1$

If we take

$$u(x,y) = S(x) + v(x,y) \tag{0.0.27}$$

where v(x,y) satisfies Laplace equation ,

(ie) $\left(\frac{\partial^2 v}{\partial x^2}\right) + \left(\frac{\partial^2 v}{\partial y^2}\right) = 0$

Since $u^{''} = -1$

$S^{''} \ + \ v^{''} = -1$

But, $v^{''} = 0$, Since v satisfies laplace equation

$\therefore S^{''} = -1$

Now $u(0,t) \ = \ 0 \ \Rightarrow S(0) = 0$

Similarly, $u(l,t) \ = \ 0 \ \Rightarrow S(l) = 0$

$\therefore \ \ we \ have$

$S^{''} = -1, S(0) = 0, S(l) = 0$

Now $S^{''} = -1 \Rightarrow S = -\frac{x^2}{2} \ + \ Bx \ + \ C$ where B and C are

unknown constants

$S(0) = 0 \Rightarrow -\frac{0^2}{2} + B(0) + C = 0 \ \Rightarrow \ \ C = 0 \Rightarrow S = -\frac{x^2}{2} \ + \ Bx$

$S(l) = 0 \Rightarrow -\frac{l^2}{2} \ + \ Bl = 0 \Rightarrow B = -\frac{l}{2} \Rightarrow S(x) = \frac{x}{2}(l - \frac{x}{2}).$

Now $u(x,0) \ = \ 0 \ \Rightarrow S(x) \ + \ v(x,0) = 0$

Similarly, $u(x,l) \ = \ 0 \ \Rightarrow S(x) \ + \ v(x,l) = 0$

$\therefore$ *we have*

$v(x,0) = \quad v(x,l) = \quad - \quad S(x) = \quad - \quad \frac{x}{2}(l - \frac{x}{2}).$

Hence,

The function "v" satisfies the Laplace equation

$$\left.\begin{aligned} &\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \\ &v(x,0) = v(x,l_1) = -\frac{x}{2}(l - \frac{x}{2}), \end{aligned}\right\} \tag{0.0.28}$$

Assume that $v(x,y) = X(x)Y(y)$.

$\Rightarrow v_x = X'Y, v_{xx} = X''Y, v_y = XY' \; v_{yy} = XY''$ where $X' = \frac{dX}{dx}, Y' = \frac{dY}{dy}$

Then eq (10) becomes

$$\begin{aligned} XY'' + X''Y &= 0 \\ (ie) \qquad \frac{-Y''}{Y} &= \frac{X''}{X} = k \; (independent\ of\ x\ and\ y) \end{aligned}$$

Suppose $k > 0$.

Then the solution of the equations

$X'' = kX$ is given by

$X(x) = c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x}$

Since $v(o,y) = v(a,y) = 0$. we get

$x(0) = c_1 + c_2 = 0$ and $X(a) = c_1 e^{\sqrt{k}a} + c_2 e^{-\sqrt{k}a} = 0$

$\Rightarrow c_1 = c_2 = 0.$

$\therefore X(x) = 0.$

$\Rightarrow v(x, y) = 0.$

$\therefore$ The given problem has no solution if $f \neq 0$.

Take $k = -\lambda^2$,

Then the solution of the equation

$X'' = -\lambda^2 X$ is given by

$X(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x).$

$$\begin{aligned} X(0) = 0 &\Rightarrow c_1 = 0 \\ X(l) = 0 &\Rightarrow c_2 \sin(\lambda l) = 0 \\ &\Rightarrow \lambda l = n\pi, n = 1, 2, \cdots \\ &\Rightarrow \lambda = \frac{n\pi}{l}, n = 1, 2, \cdots \end{aligned}$$

$\therefore X(x) = c_2 \sin \frac{n\pi x}{l},\ n = 1, 2, \cdots$

Now, $T'' = - - \lambda^2 T$

$\Rightarrow T(t) = c_3 e^{\frac{n\pi t}{l}} + c_4 e^{-\frac{n\pi t}{l}}$

$\therefore v(x, t) = c_2 \sin \frac{n\pi x}{l} [c_3 e^{\frac{n\pi t}{l}} + c_4 e^{-\frac{n\pi t}{l}}],\ n = 1, 2, \cdots$

$\Rightarrow\ v(x, t) = c_2 \sin \frac{n\pi x}{l} \left[c_5 \left(\frac{e^{\frac{n\pi t}{l}} + e^{-\frac{n\pi t}{l}}}{2} \right) + c_6 \left(\frac{e^{\frac{n\pi t}{l}} - e^{-\frac{n\pi t}{l}}}{2} \right) \right]$

$\Rightarrow \; v(x,t) = c_2 \sin\frac{n\pi x}{l}\left[c_5\left(\cosh\frac{n\pi y}{l}\right) \; + \; c_6\left(\sinh\frac{n\pi y}{l}\right)\right]$

The given problem is linear

$\therefore$ The general form of the solution is

$$v(x,y) = \sum_{n=1}^{\infty}\left(a_n\cosh\frac{n\pi y}{l} + b_n\sinh\frac{n\pi y}{l}\right)\sin\frac{n\pi x}{l} \tag{19}$$

using the BCs, we have

$-\frac{x}{2}(l-\frac{x}{2}) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}$

$-\frac{x}{2}(l-\frac{x}{2}) = \sum_{n=1}^{\infty}\left(a_n\cosh\frac{n\pi y}{l} + b_n\sinh\frac{n\pi y}{l}\right)\sin\frac{n\pi x}{l}$

$$\begin{aligned}\therefore a_n &= -\frac{2}{l}\int_0^l \frac{x}{2}(l-\frac{x}{2})\sin\frac{n\pi x}{l}dx\\ &= \begin{cases}\frac{-4l^2}{n^3\pi^3}, & n = 1,3,5,\cdots\\ 0, & n = 2,4,6,\cdots\end{cases}\end{aligned}$$

Also we have

$$a_n\cosh\frac{n\pi y}{l} + b_n\sinh\frac{n\pi y}{l} = -\frac{2}{l}\int_0^l \frac{x}{2}(l-\frac{x}{2})\sin\frac{n\pi x}{l}dx$$

wherefrom it follows that

$$b_n = \begin{cases}\frac{-4l^2}{n^3\pi^3}\left(\frac{1-\cosh\frac{n\pi y}{l}}{\sinh\frac{n\pi y}{l}}\right), & n = 1,3,5,\cdots\\ 0, & n = 2,4,6,\cdots\end{cases}$$

The solution of eq [19] is of the form

$$v(x,y) = \frac{-4l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{(2n-1)\pi x}{l}\right)}{\sinh\left(\frac{(2n-1)\pi l_1}{l}\right)}\left(\sinh\frac{(2n-1)\pi y}{l}+\sinh\frac{(2n-1)\pi(l_2-y)}{l}\right)$$

$\therefore$ The solution of the given problem is

$$u(x,y) \ = \ S(x) \ + \ v(x,y)$$

$$(ie)\ \ u(x,y) = \frac{x}{2}(l-\frac{x}{2}) - \frac{-4l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{(2n-1)\pi x}{l}\right)}{\sinh\left(\frac{(2n-1)\pi l_1}{l}\right)}\left(\sinh\frac{(2n-1)\pi y}{l}+\sinh\frac{(2n-1)\pi(l_2-y)}{l}\right)$$

(ii)

Numerical solution

GIVEN:

$$\bigtriangledown u = -1$$

$$ie \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -1$$

$$\Rightarrow \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{k^2} + \frac{\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)}{h^2} = 0 \quad (0.0.29)$$

When $x_0 = 0$, $x_f = 1$, $y_0 = 0$, $y_f = 1$, $h = 0.05$, $and\ k = 0.05$

we have ,

$J = 20\ and\ N = 20$

Since k=h=0.05, eq (0.0.29) becomes,,

$u_j^{n+1} - 4u_j^n + u_j^{n-1} + u_{j+1}^n + u_{j-1}^n = -0.0025$

then we get numerical solutions as,

Numerical Solutions

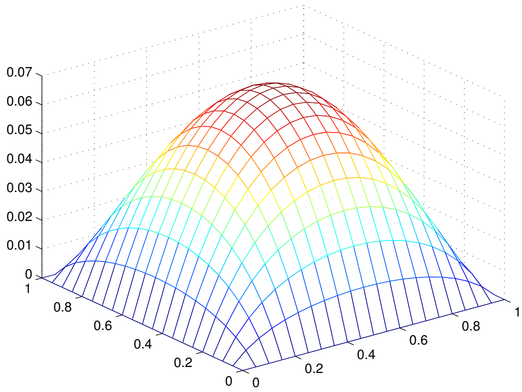
x|t	0	0.05	0.1	.	.	0.95	1
0	0	0	0	.	.	0	0
0.05	0	0.0041915863	0.007139	.	.	0.0071803509	0.0042151709
0.1	0	0.007139	0.0125455258	.	.	0.0126262426	0.0071850363
.	.	.	.	.	.	.	.
0.90	0	0.0071803509	0.0126262426	.	.	0.0126931092	0.0072184781
0.95	0	0.0042151709	0.0071850363	.	.	0.0072184781	0.0042342391
1	0	0	0.01	.	.	0	0

RELATIVE ERROR $= 1 \times e^{-4}$

MAXIMUM ITERATION $= 500$

Graphs

analytical Solution



Numerical Solution